Combinatorial Models and Network Reliability

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Reliability Models

• Model
  – Network operation
  – Network failure
  – Network repair
  – Network cost

• To determine
  – The probability that the network can carry out its intended function.
Reliability Models

• At the most basic level, the function is to provide a connection of acceptable quality and delay.
• But simple models just treat the presence or absence of a connection.
Combinatorial Models

- A graph $G=(V,E)$
  - $V$ is a set of nodes
  - $E$ is a collection of undirected or directed edges.
- An assignment of probabilities to nodes and/or edges, to indicate the probability that the node or edge operates.
Assumptions

- Each node or edge either operates or fails.
- Elements operate independently.
- Elements operate with known probabilities.

... but these aren’t true in general!
Operation

• A state is a subset of nodes and edges.
• Certain states are deemed operational; all others are failed.
• For example,
  – All connected subgraphs ("all-terminal reliability")
  – All subgraphs having an s,t-path ("two-terminal reliability")
Reliability

- The reliability measure defined is then the probability that the graph is in an operational state, when all nodes and edges operate independently with the stated probabilities.
- So is this what we are trying to compute? And if so, what does it mean?
Numbers?

- The number defined rarely captures observed performance of the network. Why?
  - The model is too simple
  - Faults interact
  - Components interact
  - Component reliabilities are estimates
  - Partial operation is not accounted for
  - Communication is more than just having a connection
Is there some theory behind these numbers?

- If our motivation in combinatorial models of reliability is to get numbers, we do not have a very compelling case.
- So what is the goal, if not just numbers...
Insight

• The capabilities of the underlying structure inform us about limitations of any actual network deployed with this structure.

• This does not ensure that we will be able to take advantage of strengths of the underlying structure.
Insight

• As a tool for network design, simple combinatorial models can provide information about
  – The best operation probability that can be hoped for
  – Structural choices that can be expected to improve the reliability
All-terminal reliability

• To illustrate these points, we consider the probability that a network is connected when nodes are perfectly reliable and edges fail, each with the same, known probability $q=1-p$.
• This is the all-terminal reliability.
Easy First Steps

- G has n nodes, m edges.
- Every state with e edges arises with probability $p^e(1-p)^{m-e}$
- We could
  - Enumerate all connected states, determining $N_e$ as the number with e edges
  - Randomly sample from all states and determine fraction of connected ones
First Steps

• But each involves treating an exponential number of states.

• In fact, determining the all-terminal reliability is \#P-complete, so we should not expect to find an efficient way to compute it.
The Reliability Polynomial

- Let \( F_i = N_{m-i} \). Then the reliability is the sum, from \( i=0 \) to \( m \), of \( F_i p^{m-i}(1-p)^i \).

- “This is not very useful in practice.”

- Why discuss it? To reveal structure, not to determine numbers!
The Reliability Polynomial

- Let $d = m-n+1$.
- Then $F_i = 0$ when $i > d$.
- $F_d$ is the number of spanning trees. This can be efficiently calculated (Kirchoff).
- Let $c$ be the size of a minimum network cut. This can be efficiently calculated by network flows.
The Reliability Polynomial

- Then $F_i$ is $m!/(i!(m-i)!)$ when $i < c$.
- And $F_c$ is $m!/(c!(m-c)!) - C_c$, where $C_c$ is the number of minimum cardinality network cuts.
- This can be efficiently calculated (Ball-Provan).
- Indeed $C_{c+k}$ can be efficiently calculated for fixed $k$ (Ramanathan-Colbourn)
The Reliability Polynomial

- But most of the coefficients cannot be efficiently calculated unless \( P = \#P \).
- Nevertheless, more reliable “means”
  - More spanning trees
  - Larger minimum network cut
  - Fewer minimum network cuts
- These are all “obvious”.
Sperner

- $F_i$ counts complements of connected subgraphs on $m-i$ edges.
- But every subset of a set counted by $F_i$ is therefore counted by $F_{i-1}$.
- In 1928 Sperner used the simple observation that the ratio of connected subgraphs to all subgraphs increases as a function of the number of edges.
Kruskal-Katona

- Kruskal and Katona improved this to determine the best lower bound on $F_i$ as a function of $F_{i+1}$ for a hereditary family of sets (equivalently, the best upper bound on $F_{i+1}$ given $F_i$).
- It can be efficiently calculated, so we get numbers...
Kruskal-Katona

• But we get more:
  • The proof employs “shifting”; we skip the details here, but it suggests that
    – High reliability means high “girth” (length of the shortest cycle)
    – High reliability means fewest cycles of length equal to the girth.

• But these are NOT proved in general!
Edge Shifting

• Let $G$ be a graph. If $\{x,y\}$, $x<y$, is an edge but $\{a,b\}$, $a<b$, is not an edge and either $b<y$ or $b=y$ and $a<x$, the edge shifting operation removes $\{x,y\}$ and adds $\{a,b\}$, provided this does not disconnect the graph.

• Conjecture: The graph having no edge shifts remaining is the least reliable $n$-vertex $m$-edge graph.
Because every connected subgraph contains a spanning tree, every maximal complement of a connected subgraph has $d$ edges.

An interval, defined by a pair of graphs $L, U$ with $U$ the complement of a spanning tree and $L$ a subgraph of $U$, consists of all subgraphs containing $L$ and contained in $U$. 
Stanley

- $H_i$ denotes the number of intervals whose lower set has size $i$
- Stanley arrived at another form of the reliability polynomial, the sum of $H_i(1-p)^i$ for $i = 0, \ldots, d$
Stanley

- By associating intervals with multisets, Stanley obtained the best lower bound on $H_i$ as a function of $H_{i+1}$ and, because these bounds can be efficiently calculated, again produced many more numbers...
Stanley

• How do we extract combinatorial structure?
• This was not well understood for 15 years, until a surprising connection was found.
Chip Firing

- Let $G=(V,E)$ be a graph
- Let $r$ be a vertex of $V$.
- A configuration $\Theta$ on $G$ is an assignment of integers to the vertices of $G$ for which $\Theta(v) \geq 0$ for $v \neq r$.

- A vertex $v \neq r$ is **ready to fire** if $\Theta(v) \geq \deg(v)$; $r$ is ready to fire if and only if no other vertex is.
Chip Firing

- *Firing* vertex $v$ means changing $\Theta$ to $\Theta'$ where
  - $\Theta(v)$ decreases by $\text{deg}(v)$
  - $\Theta(w)$ increases by the number of edges between $v$ and $w$.
- $\Theta$ is *stable* if $\Theta(v) < \text{deg}(v)$ for all $v \neq r$
Chip Firing

- A firing sequence is a sequence of configurations, each obtained from the one before by a valid firing.
- It is nontrivial if the sequence has more than one configuration.
- If a nontrivial firing sequence starts and ends with $\Theta$, then $\Theta$ is recurrent.
- Stable and recurrent $\equiv$ critical.
Chip Firing

- Now represent every vertex $v$ other than $r$ by a variable $x_v$, and form $m_{\Theta}$ as the product, over all $v$ other than $r$, of $x_v^{\deg(v)-1-\Theta(v)}$

- Treat this as a multiset by including element $x_v$ exactly $\deg(v)-1-\Theta(v)$ times, for every vertex $v$ other than $q$.

- Then let $H_i$ be the number of multisets having exactly $i$ elements, that arise from critical configurations.
Chip Firing

• Then summing $H_i(1-p)^i$ for $i = 0,\ldots,m-n+1$, and multiplying the result by $p^{n-1}$, we obtain the all-terminal reliability of $G$ when every edge operates independently with probability $p$. (Biggs, 1999; Merino, 2001)

• Note the dependence on the vertex degrees!
Pure Multicomplex

- Merino also proved that the multicomplex has the property that every maximal multiset has the same size! (It is pure.)
Structure

• This leads to the first bounds based on combinatorial complexes that use vertex degrees in a structural way, and they suggest that
  – High reliability “means” as regular as possible.

• This also remains open in general, but chip firing provides the first solid theory for the role of vertex degrees.
New Bounds

- Brown, Colbourn, and Nowakowski recently showed that, using the Clements-Lindstrom theorem, lower and upper bounds can be efficiently calculated.

- Some results follow, but the main point is that the combinatorial structure revealed by the bounds provides a theory for the role of vertex degrees.
$K_{4,4}$
$K_{6,10}$
$K_{4,5,6}$
A 21-vertex Circulant
Conclusion

- Understanding underlying combinatorial structure can both suggest and support hypotheses about the structure of reliable networks.
- Computing the numbers does not solve practical reliability problems.
- But the structure of reliable networks in general can be better understood as a result.